63-3-1

MEMORANDUM
RM-3482-PR
MARCH 1963

A NOTE ON DIFFERENTIAL APPROXIMATION AND ORTHOGONAL POLYNOMIALS

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA · CALIFORNIA

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This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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PREFACE

Part of the Project RAND research program consists of basic supporting studies. The mathematical research presented here is concerned with an analytic technique for approximating a given function by a sum of exponentials. The results that are obtained have applications in control theory, circuit synthesis, and numerical analysis.

SUMMARY

We consider a simple analytic technique for finding linear differential equations which are approximately satisfied by a given function, and thus obtain approximations to a given function by a sum of exponentials.

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A NOTE ON DIFFERENTIAL APPROXIMATION AND ORTHOGONAL POLYNOMIALS

1. INTRODUCTION

The problem of obtaining an exponential polynomial of the form $\sum_{i=1}^{N} b_i e^{\lambda_i t}$ which closely approximates a given function $f(t)$ in an interval $a \leq t \leq b$ is a problem of some difficulty if we allow both the coefficients and the exponents to be unknowns. Either of the criteria of fit,

$$(1.1) \qquad \max_{a\leq t\leq b} \left| f(t) - \sum_{i=1}^{N} b_i e^{\lambda_i t} \right|,$$

or

$$(1.2) \qquad \int_a^b \left| f(t) - \sum_{i=1}^{N} b_i e^{\lambda_i t} \right|^2 dt$$

lead to difficulties; see Lanczos [1].

In this note, we wish to consider a different way of measuring the closeness of $f(t)$ to a sum of exponentials. If $f(t)$ were a function of the form $\sum_{i=1}^{N} b_i e^{\lambda_i t}$, it would satisfy a linear differential equation of the form

$$(1.3) \qquad f^{(N)} + c_1 f^{(N-1)} + \cdots + c_N f = 0.$$

Hence, let us attempt to determine real coefficients c_i which minimize the integral

$$(1.4) \qquad \int_{-\infty}^{\infty} (f^{(N)} + c_1 f^{(N-1)} + \cdots + c_N f)^2 dt.$$

We call this <u>differential approximation</u>.

Results of the type we obtain have applications in the fields of control theory, circuit synthesis, and numerical analysis.

2. FOURIER TRANSFORMS

It is clear that we must impose some conditions on the function $f(t)$ in order to pose the problem. Assume then that $f^{(k)} \in L^2(-\infty,\infty)$ for $k = 0,1,2,\ldots,N$. Then if

$$(2.1) \qquad g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt,$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) e^{-ist} ds,$$

we have

$$(2.2) \qquad f^{(k)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-is)^k g(s) e^{-ist} ds.$$

Using the Plancherel-Parseval formula, we have

$$(2.3) \qquad \int_{-\infty}^{\infty} (f^{(N)} + c_1 f^{(N-1)} + \cdots + c_N f)^2 dt$$

$$= \int_{-\infty}^{\infty} |g(s)|^2 |c_N - isc_{N-1} + (-is)^2 c_{N-2} + \cdots|^2 ds$$

$$= \int_{-\infty}^{\infty} |g(s)|^2 \left[(c_N - s^2 c_{N-2} + \cdots)^2 + s^2 (c_{N-1} - s^2 c_{N-3} + \cdots)^2\right] ds.$$

3. ORTHOGONAL POLYNOMIALS AND MINIMIZATION

It is clear that a change of variable reduces the problem to that of finding the orthogonal polynomials associated respectively with the weight functions $|g(s)|^2$ and $s^2|g(s)|^2$. Since these orthogonal polynomials can be constructed in a systematic fashion, we have a simple way of obtaining the coefficients c_i, and further measures of the asymptotic behavior as $N \to \infty$; see Szego [2].

A direct treatment of the question of minimizing the quadratic form of (1.4) would not be simple computationally for large N (although a Schmidt orthogonalization could be used), and would not readily furnish asymptotic behavior. The technique presented above is most useful in connection with the treatment of various classical functions where $g(s)$ has a simple analytic form.

Similar results can be obtained for the case where the interval is finite or semi-infinite, but not of the same simplicity.

4. STABILITY

The study of the precise connection between the solution of the linear differential equation

$$u^{(N)} + c_1 u^{(N-1)} + \cdots + c_N u = 0, \tag{4.1}$$

and the function $f(t)$ leads to a stability question which we shall study elsewhere.

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1. Lanczos, C., Applied Analysis, Prentice Hall, Englewood Cliffs, New Jersey, 1956.

2. Szego, G., Orthogonal Polynomials, American Mathematical Society Colloquium Publications, Vol. 23, New York, 1939.

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